CHAPTER 1

ALGEBRA AND GEOMETRY

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INTRODUCTION

Algebraic demands on the safety professional can range from simple exposure calculations using conversion factors to understanding and using far more complex engineering design and exposure modeling equations. For more complex applications, the use of calculators, computer spreadsheets, and symbolic algebra software may be required. Application of complicated models is not covered in this chapter.

Algebra is the part of mathematics that uses symbolic mathematical statements to describe relationships among quantities. Letters or other symbols are often used to represent variable quantities. Letters or symbols are used to represent numbers in the following examples: $-2ax$, $5 + 67ab$, ay/z, and $x^2 = 0$. These are examples of algebraic terms, expressions, and equations, which are used to express the relationships among variables.

The basic algebraic operations presented in this section will allow the safety professional to calculate the terms of the Permissible Exposure Level (PEL) expression or those of a Threshold Limit Value (TLV) for a liquid sample and to analyze the validity of a population model, among other applications.

ALGEBRAIC TERMS

The basic operations of mathematics are addition and subtraction. When two or more objects are related by addition or subtraction, the objects are called *terms*. A term may be just a number or may consist of one or more *variables* written with letters and a numerical *coefficient*. For example, $-2ax$ contains one (1) term, and $5 + 67ab$ has two (2) terms. The term $-2ax$ has a numerical coefficient, -2, and contains the variables a and x. The numerical coefficient of the algebraic term xy is understood to be 1. Algebraic terms may contain the multiplication and division signs. Variables that are written side by side, with no symbol between, are understood to be multiplied together. In the term 89*stu,* the coefficient 89 is understood to be multiplied by the variable *s*. The variable *s* is, in turn, understood to be multiplied by the variable *t*. The variable *t* is, in turn, understood to be multiplied by the variable *u*. Terms may also contain *exponents*, expressions written above and to the right of variables, as in x^2. The value of an algebraic term is the product of the values of its coefficient, and variables, raised to the powers of their exponents. For example, the value of $3x^2$ is equal to 3 times the value of *x* times the value of *x*.

ALGEBRAIC EXPRESSIONS

Algebraic expressions show the relationships between algebraic terms. Algebraic expressions consist of one (1) or more algebraic terms and the relationships between them. The use of algebraic expressions allows the safety professional to examine the relationships that go into calculating threshold limit value (TLV), the time-weighted average (TWA), minimum sampling volume, minimum sampling rates, minimum sampling times, and other parameters that will be encountered.

An algebraic expression is a collection of one (1) or more algebraic terms to be added together, each of which must have either a positive (+) sign, which may be understood and omitted if it appears before the first term in the expression, or a negative (–) sign. The sign indicates whether an addition or a subtraction is to be done.

For example, $45xy + 89stu$, $23 + tz$, and $87/p - 67$ are algebraic expressions. The plus and minus signs in an algebraic expression separate the terms and show how they relate to each other. The plus or minus sign becomes part of the term that follows.

The expression $45xy + 89stu$ contains two (2) terms. The first term is $+ 45xy$, and the second term is $+ 89stu$. The addition sign separates the two terms. Terms that do not have a sign shown in front of them are understood to be positive.

Algebraic expressions do *not* contain an equal (=) sign.

ALGEBRAIC EQUATIONS

Algebraic equations consist of two (2) or more algebraic expressions separated by *relational operators.* Relational operators include =, ≠, <, >, ≤, and ≥, showing the relationships *equal to*, *not equal to*, *less than*, *greater than*, *less than or equal to*, and *greater than or equal to*, respectively.

We will be mostly concerned with equations in which the expression on one side is equal to (=) the expression on the other side of the equation. Such equations are like equal-arm balances. We must make sure that anything we do to the equation does not disturb the balance. Often this means we must do the same thing to both sides of the equation.

Example 1.1. A sample of chloroform is collected at a sampling rate of 1 L/min. It was determined that the required minimum sampling volume is 4.4 L. Determine the minimum sampling time.

The sampling rate, R, is given as 1 L/min. The minimum sampling volume, V, is given as 4.4 L. The minimum sampling time is:

$$t = \frac{V}{R} = \frac{4.4\text{L}}{1\frac{\text{L}}{\text{min}}} = \frac{4.4}{1\frac{1}{\text{min}}} = 4.4 \text{ min}$$

Here, the symbols V and R were replaced with their values and then the indicated operations were performed. The rules for performing such operations will be explained later in this chapter.

Example 1.2. The PEL for carbon disulfide (CS_2) is 31 mg/m^3. What is the minimum sample volume that must be collected to guarantee detection of a concentration at the TLV of 10% of the PEL with a method yielding a sensitivity of 10 µg, using the expression:

$$V = \frac{\text{AS}}{F(\text{PEL})} \tag{1.1}$$

where:

AS = analytical method sensitivity
F = fraction TLV/PEL

The PEL is given as 31 mg/m^3. The TLV is 0.10 of the PEL; that is, F is 0.10. The analytical sensitivity is:

$$\text{AS} = (10\,\mu\text{g})\left(\frac{\text{mg}}{1000\,\mu\text{g}}\right) = 0.010\text{mg}$$

The expression given is:

$$V = \frac{\text{AS}}{F(\text{PEL})}$$

Replacing the variables with their values gives:

$$= \frac{0.010 \text{ mg}}{(0.10)\left(31\frac{\text{mg}}{\text{m}^3}\right)\left(\frac{\text{m}^3}{1000 \text{ L}}\right)}$$

$$= 3.23 \text{ L } (3.2 \text{ L})$$

Example 1.3. A worker was exposed to methylene chloride ($MeCl_2$) at a concentration of 65 mg/m^3 for 1.5 hours. Later, the worker was exposed to $MeCl_2$ at a concentration of 40 mg/m^3 for 1 hour. Finally, the worker was exposed to $MeCl_2$ at a concentration of 100 mg/m^3 for 4 hours. Calculate the TWA.

The initial concentration, C_1, of $MeCl_2$ is given as 65 mg/m^3. The initial exposure time, t_1, is given as 1.5 hours. The next concentration, C_2, of $MeCl_2$ is given as 40 mg/m^3. The next exposure time, t_2, is given as 1 hour. The last concentration, C_3, of $MeCl_2$ is given as 100 mg/m^3. The last exposure time, t_3, is given as 4 hours. By definition the TWA is:

$$\text{TWA} = \frac{t_1 C_1 + t_2 C_2 + t_3 C_3}{t_1 + t_2 + t_3} \tag{1.2}$$

where t_1 through t_3 are times and C_1 through C_3 are concentrations.

$$= \frac{\left((1.5\text{h})\left(65\frac{\text{mg}}{\text{m}^3}\right) + (1\text{h})\left(40\frac{\text{mg}}{\text{m}^3}\right) + (4\text{h})\left(100\frac{\text{mg}}{\text{m}^3}\right)\right)}{1.5\text{h} + 1\text{h} + 4\text{h}}$$

$$= 82.69 \text{ mg/m}^3 \ (83 \text{ mg/m}^3)$$

SIGNIFICANT FIGURES

Significant figures show how precisely a number is known. It is important for safety professionals to handle information properly with regard to precision, accuracy, and units found in both experiments and theory. This is one of the reasons the rules for determining significant figures are presented in this chapter. Significant figures are important, for example, for the correct determination of the total length of a device, or the correct calculation of the internal volume of a furnace from measurements of the dimensions of its sides. It should be noted that the concept of significant figures applies to numbers that can be in doubt, as in measurements made or the number of items estimated. Some numbers are exactly known by definition, such as the number of feet in a mile, and others are exactly known because the items were counted by a method that is not subject to error.

Rule 1. All nonzero digits are significant. Count all digits between the leftmost and rightmost significant digits, even if those digits are zero:

- The following numbers have one (1) significant figure: 1, 5, 6, 7, 3, 9.
- The following numbers have two (2) significant figures: 93, 1.8, 6.7, 41.
- The following numbers have three (3) significant figures: 293, 205, 1.05, 4.98, 5.06, 458.

- The following numbers have four (4) significant figures: 2.869, 9.654, 1298, 9876, 3871.

Rule 2. Final zeros to the right of the decimal point are significant:

- 2.0 (2 significant figures)
- 6.10 (3 significant figures)
- 91.00 (4 significant figures)
- 123.90 (5 significant figures)

Rule 3. Zeros found between significant digits are significant:

- 5.09 (3 significant figures)
- 100.4 (4 significant figures)
- 7.0019 (5 significant figures)

Rule 4. Zeros as placeholders to the left of the first nonzero digit are *not* significant:

- 0.038 (2 significant figures)
- 0.0380 (3 significant figures)
- 0.0389 (4 significant figures)
- 0.00045896 (5 significant figures)
- 100.0 (4 significant figures)
- 1000 (1 significant figure)
- 12000 (2 significant figures)

Numbers with an Ambiguous Number of Significant Figures

Cases such as 100, 2500, and 12000, and other numbers with one or more zeros after the last nonzero digit, are ambiguous, as one cannot know how many figures are significant.

The best way to indicate significant figures is to express the number in scientific notation. In scientific notation, a number is written as a number greater than or equal to 1 and less than 10, multiplied by 10 raised to some power. For example, the number 205 is written in scientific notation as 2.05×10^2. Any number written in scientific notation has only one (1) digit to the left of the decimal point. Therefore, any zeros to the right of the decimal point are significant. For example:

- 1.0×10^2 has two significant figures
- 1.00×10^2 has three (3) significant figures

In some cases, one can use an explicit decimal point to indicate the significant figures: 100. has three (3) significant figures.

Rules for Significant Figures in Answers

In adding and subtracting numbers of different sizes, the answer can have no more significant figures than the least accurate large number involved in the operation. Report the result with only that number of significant figures.

To round the answer to the correct number of significant figures, examine the digit following the digit to be rounded. If the digit following is less than 5, the result is rounded down. If the digit following is greater than 5, the result is rounded up. If the digit following is exactly 5 the rule is to "leave it even:" If the result to be rounded has an odd digit preceding the 5, round that digit up and drop the 5. If the result has an even digit preceding the 5, just drop the 5.

Example 1.4. A device 35 mm in length is to be attached to a straight pipe with a length of 9.01 m. What is the total length of the assembly?

The length of the device is:

$$L_d = 35 \text{ mm} = 0.035 \text{ m}$$

The length of the straight pipe, L_p, is given as 9.01 m. The total length of the assembly is:

$$\begin{aligned} L &= L_d + L_p \\ &= 0.035\text{m} + 9.01\text{m} = 9.045\text{m} \end{aligned} \tag{1.3}$$

The largest number to be added has only three (3) significant figures, so the last figure in the total has to be eliminated. It should be rounded down to 9.04 m, following the "leave it even" rule.

When multiplying and dividing, report as many significant figures as in the factor with the lowest number of significant figures.

Example 1.5. The internal walls of a furnace measure 3.45 m, 2.623 m, and 1.23 m. What is the internal volume of the furnace?

The dimensions of the furnace are: $L = 3.45$ m, $w = 2.623$ m, and $h = 1.23$ m. The volume of the furnace is:

$$\begin{aligned} V &= Lwh \\ &= (3.45\text{m})(2.623\text{m})(1.23\text{m}) \\ &= 11.1307\text{m}^3 \end{aligned} \tag{1.4}$$

The result has six (6) significant figures, but it must be rounded to three (3) significant figures because L and h have only three (3) significant figures. It should be rounded down to 11.1 m^3.

Example 1.6. The length of a device was measured 3 times using different methods of measurement. The results are 2.309 mm, 2.32 mm, and 2.316 mm. What is the length of the device?

The number of measurements, n, is 3. The lengths of the device are: $L_1 = 2.309$ mm, $L_2 = 2.32$ mm, and $L_3 = 2.316$ mm. The average length of the device is:

$$\begin{aligned} L_{avg} &= \frac{L_1 + L_2 + L_3}{n} \\ &= \frac{2.309\text{mm} + 2.32\text{mm} + 2.316\text{mm}}{3} \\ &= 2.315\text{mm} \end{aligned} \tag{1.5}$$

The answer needs to be rounded to three (3) significant digits because the second length in the given data, 2.32, has only three (3) significant figures. As the last digit is exactly 5 and the last significant digit, 1, is an odd number, the answer is to be rounded upward to an even number. The best length to report for the device is 2.32 mm. This answer implies that the actual value is between 2.315 and 2.325.

TYPES OF NUMBERS

The safety engineer needs to keep in mind the distinctions between the types of numbers and what they are used to represent, because the operations allowed with different types of numbers differ.

Our numbering system is composed of *real numbers*, which can be integers (whole numbers without fractional part, which can be used for counting objects), *rational numbers*, or *irrational numbers*. A real number n is rational if it can be written as a fraction, $n = p/q$ where both p and q are integers. For example, the expression:

$$n = \left(\frac{6}{2} + \frac{2}{2}\right)^2$$

simplifies to 16, which can be written 16/1, which is a rational real number because it can be represented by a ratio of two integers. On the other hand, the numbers π and $\sqrt{5}$ are real but are not rational. They cannot be represented by any ratio of integers, and are consequently called irrational real numbers.

Because of the rules governing multiplication, any real number squared (multiplied by itself) gives a positive number: $5\times5 = -5^2 = 25$, $(-5)\times(-5) = -5^2 = -5\times-5 = 25$. There is no real number that will yield a negative number when squared. The number $\sqrt{-4}$ is called an *imaginary number* and is sometimes written as $2i$ where $i = \sqrt{-1}$. Although these numbers are called imaginary, they are useful in the calculations of some things that really do exist, such as the current through a reactive component in an AC circuit. *Complex numbers* have real and imaginary components. Imaginary and complex numbers and calculations which use them are beyond the scope of this chapter.

POLYNOMIALS

Working with *polynomials* is an important aspect of algebra. For all but the simplest problems, the safety professional will need a working knowledge of polynomials, because polynomials are one of the basic tools of algebra. Polynomials are found, for example, in linear and quadratic equations.

A polynomial is an expression with two (2) or more terms in which a finite sum of algebraic terms are present and the exponents on the variables are positive integers. The expression $3x^5 - 5x^3 + x - 24$ is a polynomial.

A *rational polynomial* can be expressed as the quotient of two (2) polynomials. The rational polynomial $p(x)$ is such that it can be expressed as the division of the polynomial $f(x)$ by the nonzero polynomial $g(x)$. The word "nonzero" is necessary because division by zero is "undefined" and thus never permitted.

Symbolically:

$$p(x) = \frac{f(x)}{g(x)}$$

The degree of a polynomial is the largest value of the exponent to which a particular variable in the expression is raised. The polynomials x, xyz, abx, and a^2x are first-degree with respect to the variable x. The polynomials x^2, x^2yz, abx^2, and $x(x + 2)$ are second-degree with respect to x.

The degree of a term is the sum of the exponents of all of the variables within the term. The term x^2yz, is a fourth-degree term; it is second degree with respect to x, while it is first degree with respect to y and to z.

The degree of a polynomial is the largest degree of all its terms. For example, the three-term polynomial $yz^3x^2 + abx^2 + x^2$ is a sixth-degree polynomial. The first term, yz^3x^2, with exponents 1, 3, and 2, is a sixth-degree term, because the sum of the exponents (2 + 1 + 3) is 6. The second term of the polynomial abx^2 with exponents 1, 1, and 2 is a fourth-degree term because the sum of the exponents (1 + 1 + 2) of the term is 4. The last term of the polynomial, x^2, is a second-degree term. The largest degree is thus 6, and the polynomial is sixth-degree. The standard form of a polynomial is written in descending order of the degrees of the terms. The term that has the largest degree is written first, the term with the next largest degree is written next, and so forth.

A polynomial may be missing terms. For example, the polynomial in the previous paragraph starts with a degree of 6, followed by a term with degree of 4. There is no term with degree 5. There is also no term of degree 3.

EQUATIONS

A glance through a few safety journals will demonstrate the importance of equations for the safety professional. Many safety-related papers, in fact, seem to contain more equations than paragraphs. It is fair to imagine that, when safety professionals communicate, there will be equations.

An equation is a statement asserting that two (2) mathematical expressions are equal.

For example, $4 + 6 = 10$, and $n = ax/p + \sqrt{5}$ and $x + 54 = 0$ are equations. The first equation has no variables; the second equation has the variables *n*, *x*, *a*, and *p* and a constant $\sqrt{5}$. The value of the variable *n* depends on the values of the variables *a*, *x*, and *p*. The variable *n* is called the dependent variable, while the variables *a*, *x*, and *p* are called independent variables. The third equation says that the value of *x* is −54.

FUNDAMENTAL ALGEBRAIC LAWS

The safety professional uses numbers and polynomials and must do so while satisfying the fundamental *algebraic laws*.

Algebraic laws state the relationships between these objects, give them meaning, and allow the development of, for example, models of pollution control, and theories to aid in the determination of terminal settling velocities of particles.

Algebra provides rules that can be used to manipulate algebraic expressions to simplify and solve equations. Among these laws are the *commutative law*, the *associative law*, and the *distributive law*.

The commutative and associative laws apply to addition and multiplication.

The commutative law for addition of two (2) variables *a* and *b* can be expressed as:

$$a + b = b + a \tag{1.6}$$

The commutative law of multiplication of two (2) variables *a* and *b* can be expressed as:

$$ab = ba \tag{1.7}$$

The commutative law for addition and multiplication says that the order in which the variables are added or multiplied makes no difference in the result.

The associative law for the addition of three (3) variables a, b, and c can be expressed as:

$$a + (b + c) = (a + b) + c \tag{1.8}$$

The associative law of multiplication of three (3) variables a, b, and c can be expressed as:

$$a(bc) = (ab)c \tag{1.9}$$

It may be interesting to note that for matrices, which are beyond the scope of this book, the associative law of multiplication does hold, but the commutative law does not.

In summary, adding or multiplying a grouping of variables produces the same result regardless of the order or the grouping of variables.

The distributive law of three (3) variables a, b, and c can be expressed as:

$$a(b + c) = ab + ac \tag{1.10}$$

In summary, adding several variables and multiplying the result is equivalent to multiplying the single variable with each of the variables in the grouping, then adding the result.

Notice that multiplication is also distributive over subtraction. For example, (2) (5 – 2) = (2×5) – (2×2) = 10 – 4 = 6. These laws are used to expand algebraic expressions.

Example 1.7. The surface area of a rectangular pit is given by the expression $(3)(3a + 4)$ Simplify the expression. The expression given is:

$$(3)(3a + 4) = (3)(3a) + (3)(4) = 9a + 12$$

Example 1.8. The heat capacity of a mixture of two (2) components, as a correlation of the heat capacities of the substances, is given as $6.1a + 4.0b + 7.5a + 5.8b$. Simplify the expression. The expression given is:

$$6.1a + 4.0b + 7.5a + 5.8b = (6.1a + 7.5a) + (4.0b + 5.8b) = 13.6a + 9.8b$$

Example 1.9. A worker is exposed to a solid organic hazardous pollutant composed of particles having an average diameter of 0.001 μm and a density of 0.86 g/cm^3. The pollutant has been accidentally released into the air. In this particular work environment, the pollutant particles settle out of the air at a terminal settling velocity V_t following Stokes's Law:

$$V_S = \frac{(\rho_A - \rho)gd^2}{\eta} \tag{1.11}$$

where:

V_S = terminal settling velocity

ρ = density of particle

ρ_A = constant equal to 0.00117 g/cm^3 (the density of air)

g = constant equal to 981 cm/sec^2 (the acceleration due to gravity)

d = diameter of particle

η = constant equal to 1.73×10^{-5} N · sec/m^2 (representing the viscosity of air)

Calculate the terminal settling velocity of the particles.

Note: In the problem above, "g" is not the same as "*g*". The first is the abbreviation for grams, the other is the gravitational constant. In algebra, small differences in how a letter is printed can make it mean something completely different.

The average diameter of the particles is

$$d = 0.001\,\mu\text{m} = 0.001 \times 10^{-4}\ \text{cm} = 1 \times 10^{-7}\ \text{cm}$$

For the particle Stokes's Law is given as:

$$V_S = \frac{(\rho - \rho_A) g d^2}{\eta}$$

$$= \frac{\left(0.86\frac{\text{g}}{\text{cm}^3} - 0.00117\frac{\text{g}}{\text{cm}^3}\right)\left(981\frac{\text{cm}}{\text{sec}^2}\right)(1 \times 10^{-7}\ \text{cm})^2}{\left(1.73 \times 10^{-5}\ \frac{\text{N} \cdot \text{sec}}{\text{m}^2}\right)\left(\frac{\frac{\text{kg} \cdot \text{m}}{\text{sec}^2}}{\text{N}}\right)\left(100\frac{\text{g}}{\text{kg}}\right)}$$

$$= 4.867 \times 10^{-10}\text{m/sec} \approx 5 \times 10^{-10}\text{m/sec}$$

The answer needs to be rounded to one (1) significant digit because the particle size in the given data, 0.001 m, has only one (1) significant figure.

Example 1.10. An asbestos sample was taken using a filter having a usable filter area of 385 mm^2, for 150 min, at a constant rate of 2.5 L/min. The filter, having a graticule diameter of 100 μm, was analyzed using NIOSH Method 7400. The analysis yielded an average measured fiber count per field of 12, while the blank filters yielded 0.020 fibers per field. The calculation of the air concentration of an asbestos sample may be performed using the following equation:

$$C = \frac{(C_S - C_b)A_S}{A_f\, tR} \tag{1.12}$$

where:

C = fiber concentration

C_S = average measured fiber count per field

C_b = average blank fiber count per field

A_S = surface area of filter

A_f = area of the field

t = time of sampling

R = sampling rate

Calculate the concentration of asbestos fibers in the air sample.

The usable filter area, A_s, is given as 385 μm². The time over which the sampling was performed, t, is given as 150 min. The sampling rate, R, is given as 2.5 L/min. The filter diameter of the graticule, d, is given as 100. μm. The radius of the graticule is:

$$r = \frac{d}{2} = \frac{(100.\,\mu\text{m})\left(\dfrac{\text{mm}}{1000\,\mu\text{m}}\right)}{2} = 0.0500 \text{ mm}$$

The cross-sectional area of the graticule is:

$$A_f = \pi r^2 = \pi(0.0500\text{mm})^2 = 0.00785\ \frac{\text{mm}^2}{\text{field}}$$

The average measured fiber count, C_s, is given as 12 fibers/field. The average blank fiber count, C_b, is given as 0.020 fibers per field. The concentration is given as:

$$C = \frac{(C_S - C_b)A_S}{A_f\, tR}$$

$$= \frac{\left(12\dfrac{\text{fibers}}{\text{field}} - 0.020\dfrac{\text{fibers}}{\text{field}}\right)(385\ \text{mm}^2)}{\left(0.00785\dfrac{\text{mm}^2}{\text{field}}\right)(150\ \text{min})\left(2.5\dfrac{\text{L}}{\text{min}}\right)\left(1000\dfrac{\text{cm}^3}{\text{L}}\right)}$$

$$= 1.567\ \text{fibers/cm}^3\ (1.6\ \text{fibers/cm}^3)$$

ROOTS OF QUADRATIC EQUATIONS

All second degree equations can be arranged into a standard format call the *quadratic equation.* The roots of a quadratic equation are the values of the variable that satisfy the equation. The safety professional applies the roots of quadratic equations, for example, to find values of contaminant patterns, determine adverse effects of materials used in the workplace, or to calculate the length of a container.

The standard form quadratic equation $ax^2 + bx + c = 0$ has two (2) solutions:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Often this is written as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1.13)$$

The expression under the square root sign, $b^2 - 4ac$, is called the *discriminant.* The discriminant shows the nature of the roots. When the discriminant is zero, the quadratic equation has one (1) root.

For example, the quadratic equation $5x^2 + 5x + 1.25 = 0$, with $a = 5$, $b = 5$, and $c = 1.25$, has only one (1) root, because the discriminant $b^2 - 4ac = 5^2 - (4)(5)(1.25)$ is zero.

If the discriminant of a quadratic equation is positive, the equation has two (2) distinct real roots, x_1 and x_2. For example for the quadratic equation, $5x^2 + 26.85x + 2.65 = 0$, $a = 5$, $b = 26.85$, and $c = 2.65$. For this equation the discriminant is:

$$\begin{aligned} b^2 - 4ac &= 26.85^2 - (4)(5)(2.65) \\ &= 667.9 \end{aligned}$$

This quadratic equation has two (2) real solutions. The solutions are:

$$\begin{aligned} x_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-26.85 + \sqrt{26.85^2 - (4)(5)(2.65)}}{(2)(5)} \\ &= -0.1006 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-26.85 + \sqrt{26.85^2 - (4)(5)(2.65)}}{(2)(5)} \\ &= -5.269 \end{aligned}$$

Example 1.11. The length of a container in meters is given by the quadratic equation $6.32x^2 - 7.54x - 5.98 = 0$. What is the length of the container?

For this equation, $a = 6.32$, $b = -7.54$, and $c = -5.98$. The roots are given by:

$$\begin{aligned} x_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7.54) + \sqrt{(-7.54)^2 - 4(6.32)(-5.98)}}{(2)(6.32)} \\ &= 1.74 \end{aligned}$$

and

$$\begin{aligned} x_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7.54) + \sqrt{(-7.54)^2 - 4(6.32)(-5.98)}}{(2)(6.32)} \\ &= -0.544 \end{aligned}$$

The length of the container must be a positive number, so the correct answer is 1.74 m.

Example 1.12. The following expression $56.3x^2 + 4.65x - 9.00 = 2.37x^2 + 64.1$, is obtained after analyzing the flow rate of a piping system. What is the value of x?
The equation given, $56.3x^2 + 4.65x - 9.00 = 2.37x^2 + 64.1$, simplifies to:

$$53.93x^2 = 4.65x - 73.1 = 0$$

For this equation:

$a = 53.93$
$b = 4.65$
$c = -73.1$

The roots are:

$$\begin{aligned} x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(4.65) \pm \sqrt{(4.65)^2 - 4(53.93)(-73.1)}}{(2)(53.93)} \\ &= 1.1219 \text{ and } -1.208 \end{aligned}$$

Because the flow rate is a positive number, the value of x is 1.12.

Example 1.13. A quantitative study of cold stress caused by climate factors, considering air temperature, wind speed, and wind chill, proposed the following expression:

$$0.348x^2 + 2.360x = -4.000$$

Find the roots of the expression. The quadratic equation given is:

$$0.348x^2 + 2.360x = -4.000$$

Rearranged to:

$$0.348x^2 + 2.360x + 4.000 = 0$$

The coefficients have the following values:

$a = 0.348$
$b = 2.36$
$c = 4.000$

The roots are:

$$\begin{aligned} x_1 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(2.36) \pm \sqrt{(2.360)^2 - 4(0.348)(4.000)}}{(2)(0.348)} \\ &= -3.45 \text{ and } -3.33 \end{aligned}$$

RULES FOR EXPONENTS AND RADICALS

The safety professional must manipulate radicals and exponents without error. Many of the equations in the safety professional field involve exponents. The rules of exponents and radicals will allow the safety professional to calculate, for example, the return activated-sludge flow, signal responses, concentrations of pollutants, rate of reactions, removal efficiencies, and mass rates.

In the expression b^n, b is known as the base and n is the exponent. The exponent or power, n, of b is written as the superscript. The base b is said to be in the exponential form. With real numbers, a, b, m, and n, the following rules apply. Any base to a power of zero is 1:

$$b^0 = 1 \tag{1.14}$$

Any base to the power of 1 is equal to the base:

$$b^1 = b \tag{1.15}$$

Any base, with a value different from zero, and with a negative exponent is equal to the inverse of the same base raised to the same magnitude positive exponent:

$$b^{-n} = \frac{1}{b^n} = \left(\frac{1}{b}\right)^n \tag{1.16}$$

Example 1.14. The return activated-sludge flow as a percentage of the influent flow is 3.3^{-2}. Give a numerical value for the return activated-sludge flow.
The expression given is:

$$3.3^{-2}$$

Simplifying gives:

$$3.3^{-2} = \frac{1}{3.3^2} = \left(\frac{1}{3.3}\right)^2 = 0.092$$

To multiply identical bases, add the exponents:

$$a^m a^n = a^{m+n}$$

To divide identical bases, subtract the exponents:

$$\frac{a^m}{a^n} = a^{m-n}$$

To raise a product to a power, raise each component of the product to the power:

$$(ab)^n = a^n b^n$$

For example:

$$(3b)^2 = 3^2 b^2 = 9b^2$$

To divide different bases, raise each base to the power and divide the results:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Example 1.15. The response signal of a pneumatic controller is 2 times the amplitude of the signal, x, to the power of 2. The response signal of a second pneumatic controller is equal to the amplitude of the signal to the power of 3. The combined response signal of the two (2) pneumatic controllers is equal to the product of the individual response signals. Determine the combined response signal. The response signal of the first pneumatic controller is:

$$S_1 = 2x^2$$

The response signal of the second pneumatic controller is:

$$S_2 = x^3$$

The combined response signal is the product of the individual response signals:

$$S = S_1 S_2 = 2x^2 x^3 = 2x^{2+3} = 2x^5$$

Example 1.16. A rate of reaction of an aqueous solution of 1,3-butadiene is given as $R = (ab)^2$. Determine the rate of reaction when $a = 3.4$ g/L and $b = 6.98$ g/L. The rate of reaction is given as:

$$R = (ab)^2$$

Expanding the preceding expression and simplifying gives:

$$R = (ab)^2 = \left(\left(3.4\frac{\text{g}}{\text{L}}\right)^2\left(6.98\frac{\text{g}}{\text{L}}\right)\right)^2 = \left(3.4\frac{\text{g}}{\text{L}}\right)^2\left(6.98\frac{\text{g}}{\text{L}}\right)^2$$

$$= \left(11.56\frac{\text{g}^2}{\text{L}^2}\right) = \left(48.72\frac{\text{g}^2}{\text{L}^2}\right)$$

$$= 563.2078 \text{ g}^4/\text{L}^4 \ (560 \text{ g}^4/\text{L}^4)$$

Example 1.17. A thermal oxidizer is removing carbon monoxide and nitrogen monoxide from a gaseous waste stream. The removal efficiency is the square root of the ratio of the concentrations of carbon monoxide to nitrogen monoxide in the gaseous stream. Determine the thermal oxidizer removal efficiency when the concentration of carbon monoxide is 2.22 mg/m^3 and that of nitrogen monoxide is 7.54 mg/m^3.
The concentration of carbon monoxide, C_1, is given as 2.22 mg/m^3. The concentration of nitrogen oxide, C_2, is given as 7.54 mg/m^3. The removal efficiency is:

$$E = \sqrt{\frac{C_1}{C_2}} = \left(\frac{C_1}{C_2}\right)^{0.5} = \frac{\left(2.22\frac{\text{mg}}{\text{m}^3}\right)^{0.5}}{\left(7.54\frac{\text{mg}}{\text{m}^3}\right)^{0.5}}$$

$$= 0.5426$$

This should be rounded to 0.543 because all of the measurements have three (3) significant figures.

LOGARITHMS

The safety professional may use *logarithms* to calculate, for example, the time it takes for some chemical reactions to reach certain levels or for some microorganisms to grow; the stability of various materials, such as steel; average production rates; the efficiency of control devices; mass flow; volumetric flow; and production rates. Logarithms play an important role anywhere that there are exponentials.

In many safety fields, the use of an exponent or a logarithm can make it easier to see the correlation of events, because an exponent or logarithm may convert a complicated-looking correlation into a linear expression, simplifying the problem.

In addition, where exponents exist in an equation, the safety professional may use logarithms in solving some of the variables related to the exponents.

The logarithm, y, of a number, a, with respect to a base, b, is the exponent to which one must raise b to obtain a. As 1 to any power is equal to 1, it may not be used as a base for a logarithm. For a logarithm to be defined, the base b must be different from 1. The logarithm of zero ($\log_b(0)0$) is undefined for all bases:

$$y = \log_a a \text{ means that } a = b^y \tag{1.17}$$

Logarithms with respect to the base 10 are called *common logarithms*. The phrase "common logarithm of" is written as "log" in mathematical expressions. The words "log" and "logs" are often used informally as abbreviations for "logarithm" and "logarithms." Common logs are useful because they indicate the order of magnitude of the corresponding numbers. For example, log 0 equals 10^0 or 1; log 1 equals 10^1 or 10; log 2 equals 10^2 or 100.

Another kind of logarithm, called the *natural* or *neperian logarithm*, often appears in equations developed using calculus. The base of the natural or neperian logarithms is the Euler number e, an irrational number that is 2.718 when given to four (4) significant figures. The natural logarithm is written as ln, for example, $\ln x/\ln 10$. The natural logs are related to common logs in the following ways, the common logarithm of the number x is:

$$\log x = \ln x \log e = \ln x \log 2.718 = 0.4343 \ln x \tag{1.18}$$

and

$$\ln x = \log x \ln 10 = 2.3026 \log x \tag{1.19}$$

Both the common and natural logarithms are displayed in scientific calculators. There are three (3) usual rules for operations with logarithms.

Rule 1. Since logarithms represent exponents, the logarithm of a product is the sum of the logarithms of the factors:

$$\log ab = \log a + \log b \tag{1.20}$$

It should be noted that logarithms and exponentials can only be applied to pure numbers. Units must be removed prior to actually evaluating the logarithm or exponential. This means that constants used in logarithmic and exponential expressions must include the proper units in order to cancel out the units on the variables.

Example 1.18. The rate of growth, N, of a bacteria on a filter depends on the time, t, after the filter has been replaced and has been determined to be given by $N = \log(3.54t/\text{min})$. What is the rate of growth of the bacteria 25 minutes after changing the filter? The time, t, is given as 25 min. The expression given is:

$$N = \log\left(\frac{3.54}{\text{min}}t\right)$$

Expanding the preceding logarithmic expression, rearranging, and replacing gives:

$$\begin{aligned} N &= \log\left(3.54\frac{t}{\text{min}}\right) = \log 3.54 + \log\frac{t}{\text{min}} \\ &= 0.5490 + \log\frac{t}{\text{min}} = 0.5490 + \log\frac{25\,\text{min}}{\text{min}} \\ &= 0.5490 + \log 25 = 0.5490 + 1.398 \\ &= 1.947 \end{aligned}$$

Rule 2. As logarithms represent exponents, the logarithm of a ratio of two numbers is the difference of the logarithms of the numerator and the denominator:

$$\log\frac{a}{b} = \log a - \log b \tag{1.21}$$

Example 1.19. The average rate of production, R_1, of a first production line is 22.67 articles per min, while the average rate of production, R_2, of a second production line is of 12.75 articles per min. The net average production rate, R, of the two production lines is given by the expression $R = \log R_1/R_2$. What is the average rate of production of the combined lines?

The average rate of production of the first line, R_1, is given as 22.67. The average rate of production of the second line, R_2, is given as 12.75. The expression of the combined production lines is given as:

$$R = \log\frac{R_1}{R_2}$$

Expanding the preceding expression and replacing gives:

$$R = \log\frac{R_1}{R_2} = \log\frac{22.67}{12.75} = \log 22.67 - \log 12.75$$
$$= 1.3555 - 1.1055 = 0.2499$$

Example 1.20. Two (2) independent control devices in series control a pollutant in a waste stream. Under a determined set of conditions, the efficiency, E_1, of a control device is 0.678, while the efficiency, E_2, of a second control device is 0.876. The resultant efficiency, E, of the two (2) control devices is given by the expression, $E = \log 3E_1/0.69E_2$. What is the combined efficiency of the control devices?

The efficiency, E_1, of the first control device is given as 0.678. The efficiency, E_2, of the second control device is given as 0.876. The resultant efficiency is given as:

$$E = \frac{3E_1}{0.69E_2}$$

Expanding the preceding expression and replacing gives:

$$E = \log\frac{3E_1}{0.69E_2} = \log 3E_1 - \log 0.69E_2$$
$$= \log 3 + \log E_1 - \log 0.69 - \log E_2$$
$$= \log 3 + \log 0.678 - \log 0.69 - \log 0.876$$
$$= 0.4771 + (-0.1688) - (-0.1612) - (-0.0005750)$$
$$= 0.527 = 0.53$$

Rule 3. As logarithms represent exponents, the logarithm of a number raised to a power is the power times the logarithm of the number:

$$\log a^n = n \log a \quad (1.22)$$

Example 1.21. The concentration, R, in pounds per minute, of one of the components in the outlet stream of a reactor is given by $R = \log 3.96^{2.0\text{lb/min}}$. What is the rate of production of this component? The expression is given as:

$$R = \log 3.96^{2.0\text{lb/min}}$$

Expanding the preceding logarithmic expression and solving gives:

$$R = \log 3.96^{2.0\,\text{lb/min}} = \left(2.0\frac{\text{lb}}{\text{min}}\right)\log 3.96$$

$$= \left(2.0\frac{\text{lb}}{\text{min}}\right)(0.5977)$$

$$= 1.1954\ \text{lb/min} = 1.2\ \text{lb/min}$$

SIMULTANEOUS LINEAR EQUATIONS

A *linear equation* is one that contains no term with a degree higher than 1. It is called linear because it can be shown as a straight line on a graph. The safety professional will find that many problems in statistical analysis and applied mathematics are linear in nature. Examples of these systems are computation of the stress in a simple truss and the calculation of the acceleration of a rocket, given the upward velocities at different times. In many cases the safety professional may be able convert problems that are not linear into linear equations. The safety professional will use *simultaneous linear equations* to calculate numbers of products in an assembly line, the value of parameters involving production rates, temperature distributions, concentration of pollutants, rate of electricity production in a power plant, and many parameters involving safety conditions.

Equations often arise in which there is more than one unknown quantity or variable. Solving such problems requires more than one equation to be combined in a system of simultaneous linear equations. For example, the two (2) linear equations $x + 3y = 22$ and $3x - 2y = 22$ each contain the same two (2) unknowns, x and y. To solve this system of equations, values of x and y must be found to satisfy both equations simultaneously, and the two equations are called simultaneous equations. Solving a system of linear equations is equivalent to finding the location of the point of intersection of straight lines.

The solution of the two preceding equations is $x = 8.1818$ and $y = 1.2727$. Simultaneous equations can be solved by the method of *elimination*, *substitution*, or *graphical plotting*.

Many times the number of variables in the system of simultaneous equations is greater than 2. In situations where there are more than three (3) variables in the linear system, the graphical method is not applicable. Finding the solution to these more general linear systems is best done by the use of matrix algebra.

In solving a system by elimination, the objective of the each step is to remove one of the unknowns or variables.

One rule in algebra allows the addition or subtraction of the same number to both sides of the equation. Another rule allows both sides of any equation to be multiplied or divided by the same nonzero number without altering the solution of the equation.

For example, both sides of the equation $x + 7y = -2.4$ may be multiplied by 6. The result is:

$$6x + 42y = -14.4$$

This equation and the original one are equivalent.

Because *all* the equations in the system must be satisfied simultaneously and because the left and right sides of an equation are equal, subtracting one equation from another can be used to eliminate a variable. By multiplying both sides of one of the equations by the same number, the coefficients can be modified so that the coefficient of a variable is the same as in another equation in the system. Then, the left side of one equation can be subtracted from the left side of the second equation, while the right side of the first equation is subtracted from the right side of the second equation. This process can be continued until all unknowns are eliminated, leaving the value of one of the unknowns. That value can then be substituted back into one of the previous equations to give another variable's value, and so on, until all unknowns are eliminated or known.

Example 1.22. In a factory, the finished product requires two components. The first component is produced in machine A, and the second component in machine B. Machine A operates such that the number of components produced depending on the amount of two raw materials, x and y, is given by the expression:

$$2x - 7y = 160$$

Machine B operates according to the expression:

$$3x - 8y = 10$$

How much material x is expended producing 170 items? The equations given are:

$$2x - 7y = 160 \tag{1.23}$$

$$3x - 8y = 10 \tag{1.24}$$

Notice that a total of 170 items are produced, according to the given equations.

To eliminate the variable y, modify the equations so that the coefficients, given as 8 and 7, of the variable y are the same. This can be achieved multiplying both sides of Equation 1.23 by 8 and Equation 1.24 by 7. Multiplying both sides of Equation 1.23 by 8 gives:

$$\begin{aligned}(2)(8)x - (7)(8)y &= 160(8)\\ 16x - 56y &= 1280\end{aligned} \tag{1.25}$$

Multiplying both sides of Equation 1.22 by 7 gives:

$$\begin{aligned}(3)(7)x - (8)(7)y &= (10)(7)\\ 21x - 56y &= 70\end{aligned} \tag{1.26}$$

Subtracting Equation 1.26 from Equation 1.25 gives:

$$5x = -1210$$

$$x = -242 \text{ (242 used)}$$

The value of x that satisfies the system of equations is 242.

Example 1.23. A pharmaceutical company produces two (2) enzymes simultaneously. The enzymes' production is controlled by two (2) parameters, y and z. The production of the first enzyme is given by:

$$3.52y + 8.2z = 50.45$$

The production of the second enzyme is given by:

$$22.3y - 9.3z = 106$$

Determine the value of the variable z that satisfies the simultaneous linear equations system.

The equations given are:

$$3.52y + 8.2z = 50.45 \tag{1.27}$$

$$22.3y - 9.3z = 106 \tag{1.28}$$

To eliminate the variable y, modify the equations so that the coefficients, 3.52 and 22.3 of the variable y are the same with opposite signs. This can be achieved multiplying both sides of Equation 1.27 by 22.3 and Equation 1.28 by –3.52. Multiplying both sides of Equation 1.27 by 22.3 gives:

$$\begin{aligned}(3.52)(22.3)y + (8.2)(22.3)z &= (50.45)(22.3)\\ 78.496y + 182.86z &= 1125.035\end{aligned} \tag{1.29}$$

Multiplying both sides of Equation 1.28 by –3.52 gives:

$$\begin{aligned}(22.3)(-3.52)y + (9.3)(-3.52)z &= (106)(-3.52)\\ -78.496y + 32.736z &= -373.12\end{aligned} \tag{1.30}$$

Adding Equation 1.29 and Equation 1.30 gives:

$$215.596z = 751.915$$

$$z = \frac{751.915}{215.596} = 3.4876 \text{ (3.5)}$$

The solution is 3.5.

Example 1.24. At a temperature of 1°C, a porous substance absorbs 0.5 ppm of a pollutant. At a temperature of –3°C the same absorbent absorbs 45 ppm of the pollutant.

The concentration (in ppm) of the pollutant in the absorbent is known to be directly proportional to the temperature (in degrees Celsius). Determine the equation of the straight line that describes the concentration of pollutant versus temperature.
Define temperature t_1 as 1°C. At this temperature the concentration, y_1, is given as 0.5 ppm. At temperature t_2 of −3°C the concentration, y_2, is given as 45 ppm. The first pair of coordinates is (1°C, 0.5 ppm), and the second pair of coordinates is (−3°C, 45 ppm). The relationship between the concentration and the temperature is given as directly proportional or linear, which means there is a straight line that passes through these two (2) points. The equation of a straight line is:

$$y = mx + b \tag{1.31}$$

Because the first pair of coordinates is 1°C, 0.5 ppm, the values of x and y are 1°C and 0.5 ppm, respectively. Replacing in Equation 1.31 gives:

$$0.5 \text{ ppm} = m(1°\text{C}) + b$$

Rearranging produces:

$$m(1°\text{C}) + b = 0.5 \text{ ppm} \tag{1.32}$$

For the second pair of coordinates, (−3°C, 45 ppm), the values of x and y are −3°C and 45 ppm, respectively. Replacing in Equation 1.31 gives:

$$45 \text{ ppm} = m(3°\text{C}) + b$$

Rearranging produces:

$$m(3°\text{C}) + b = 45 \text{ ppm} \tag{1.33}$$

Equations 1.32 and 1.33 are a system of simultaneous linear equations with variables b and m. Because the coefficients of the variable b in Equation 1.32 and Equation 1.33 are equal, elimination of the variable b is achieved by subtracting Equation 1.32 from 1.33, which yields:

$$(-4°\text{C})\,m = 44.5 \text{ ppm} \tag{1.34}$$

Solving for m:

$$m = \frac{-44.5 \text{ ppm}}{4°\text{C}} = -11.125 \text{ ppm/°C}$$

Solving Equation 1.32 for the intercept of the straight line, b, gives $b = 11.625$ ppm. The equation of the straight line is:

$$y = \left(-11.125 \frac{\text{ppm}}{°\text{C}}\right) x + 11.625 \text{ ppm}$$

When solving a system by substitution, the objective is to rearrange one equation so that one variable is expressed as a combination of the other variables. Given a system of simultaneous linear equations, substitution of one of the variables can be achieved by modifying the equations so that one of the variables is independent. Substitution of the chosen variable in the remaining equations eliminates that variable.

Example 1.25. The concentration of a certain toxic material is dependent on two (2) variables, x and y. When the substance is in the gaseous phase as it reaches the site of reaction, these two variables are related by the following expression:

$$x + 3.34y = -53.0$$

The same two variables are related by the expression:

$$x - 23.4y = 65.2$$

when the toxic material reaching the site of reaction is in the liquid phase, and all other experimental conditions are the same. Determine the value of the variable x. The system given is:

$$x + 3.34y = -53.0 \quad (1.35)$$

$$x + 23.4y = 65.2 \quad (1.36)$$

Rearranging Equation 1.35 for the variable x gives:

$$x = -53.0 - 3.34y \quad (1.37)$$

Using the method of substitution, replacing Equation 1.37 in Equation 1.36 gives:

$$-53.0 - 3.34y - 23.4y = 65.2$$

Rearranging and solving for the variable y gives:

$$y = \frac{65.2 + 53.0}{-3.34 - 23.4} = -4.4203 = -4.42$$

Substitution of this value in Equation 1.35 and solving for x gives:

$$x = -53.0 - (3.34)(-4.4203)$$
$$= -38.2362 = -38.2$$

SYSTEMS OF EQUATIONS WITH NO SOLUTION

In some instances, the safety professional will be asked to analyze correlations for which solutions do not exist.

On occasion a pair of simultaneous linear equations has no solution. This system is made of inconsistent equations. The two equations actually represent two (2) parallel lines, so there is no point of intersection. For example, the pair of simultaneous equations, $5y - 3z = -1$ and $-2.5y + 1.5z = -2.5$, has no solution. In trying to eliminate y, first multiply the second equation by 2:

$$2.5(2y) + 1.5(2)z = -2.5(2)$$
$$-5y + 3z = -5$$

Then add the two equations together:

$$5y - 3z = -1$$
$$-5y + 3z = -5$$
$$0y + 0z = -6$$
$$0 = -6 \text{ (can never be true)}$$

Likewise the system $8A + 1.5B = -6$ and $-16A - 3B = 0$ has no solution.

SYSTEMS OF EQUATIONS WITH AN INFINITE NUMBER OF SOLUTIONS

A safety professional may sometimes find that the correlation under study has an infinite number of solutions. Such a system is said to be a *redundant system* or *overspecified.*

Some pairs of simultaneous linear equations may yield multiple solutions. For example, the equations $2.2A + 1.04B = 5.68$ and $1.1A + 0.52B = 2.84$ have an infinite number of solutions. These equations are identical. The first equation is the same as the second equation multiplied by 2. The second equation is redundant because it reduces to the first one. A linear system of equations with redundant equations has an infinite number of solutions.

FUNCTIONS

An algebraic expression that can have only one (1) value for each possible combination of values of the variables it contains is said to be a *function* of those variables. For example, $87/p-67$ is a function of one (1) variable, p, whereas $45xy + 89stu$ is a function of five (5) variables, s, t, u, x, and y. The notation $f(x)$ represents a function of the variable x, while $f(s, t, u)$ indicates a function of three variables s, t, and u. A statement such as $f(s, t, u) = 89stu$ is a definition of the function f rather than an equation that can be solved to find some variable.

There are also functions that are not algebraic expressions. For example, the amount of money in a bank checking account is a function of time, t, because at any given time there is only one (1) value for the amount of money, but there is no algebraic expression that can be used to compute the amount of money directly from the time alone.

A function $f(x)$ is continuous if arbitrary small changes in the input produce small changes in the output. For example, $y = x^2$ is a continuous function. Another continuous function is the average temperature of air at specific heights in a tank as the tank is heated at the bottom.

When small changes in the input to a function produce jumps in the output, the function is said to be discontinuous or have a *discontinuity*. For example, if $f(t)$ is defined as the amount of money in a checking account at a determined time, t, the function $f(t)$ jumps whenever there is a deposit, withdrawal or a check clears.

The values that the variables in a function can take are called the *domain* of that function. Some functions, such as $f(x) = x^2$, can take any real values of x. On the other hand, the function $f(x) = x^2 - 1/x - 1$ is undefined when $x = 1$ because the denominator would be zero and division by zero is undefined. The value $x = 1$ is said to be not in the function domain.

LIMITS

The *limit*, L, of a function $f(x)$ is the value that the function approaches as x approaches a specific value. In many cases the safety professional will need to determine limits to calculate, for example, acceptable leak rates, the detection ranges of many hazardous materials, maximum power in an electric production plant, and maximum or minimum production rates in a specific plant.

For example, the limit of $f(x) = 3x + 7$ is 7 when x approaches zero and is 13 when x approaches 2. The limit of a constant is a constant.

Limits can represent how a function behaves around values outside its domain. For example, $x = 1$ is not in the domain of the function $f(x) = x^2 - 1/x - 1$ because division by zero is undefined. When $x = 0.99$, $f(x)$ is 1.99. When x is 0.999, $f(x)$ is 1.999, and when $x = 0.9999$, $f(x)$ is 1.9999. One can see that $f(x)$ approaches the value of 2 when x approaches the value of 1. The limit, L, of $f(x)$ is 2 when x approaches 1, expressed as:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

This limit may be verified by simplifying the expression that defines $f(x)$:

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1) + (x + 1)}{x - 1} = x + 1$$

Therefore, at the point of discontinuity, when x = 1, the value of the function is 2.

Example 1.26. A rupture disk, placed in a pipe transporting a gaseous hazardous waste, is designed for a maximum pressure of the hazardous waste in the pipe of 2.01 atm. The design equation is the limit of:

$$\frac{x^3 - 27.5 \text{ atm}^3}{x^2 - 9.03 \text{ atm}^2}$$

when x approaches 2.01 atm. Determine the value of this limit. Substituting the value $x =$ 2.01 atm into the expression gives:

$$\frac{x^3 - 27.5\ \text{atm}^3}{x^2 - 9.03\ \text{atm}^2} = \frac{(2.01\ \text{atm})^3 - 27.5\ \text{atm}}{(2.01\ \text{atm})^2 - 9.03\ \text{atm}}$$
$$= 3.8837\ \text{atm}\ (3.88\ \text{atm})$$

Example 1.27. The maximum power generated by a turbine in an electrical plant is determined by the limit of:

$$\frac{\dfrac{x^2}{\text{sec}\cdot\text{joule}} + \dfrac{x}{\text{sec}} - 365\dfrac{\text{joule}}{\text{sec}}}{4.0\ \dfrac{x}{\text{joule}} + 34}$$

Calculate the maximum power as x approaches 2300 joules.
Substituting the value $x = 2300$ joules into the expression gives:

$$\lim_{x \to 2300\cdot\text{joule}} \frac{\dfrac{x^2}{\text{sec}\cdot\text{joule}} + \dfrac{x}{\text{sec}} - 365\dfrac{\text{joule}}{\text{sec}}}{4.0\ \dfrac{x}{\text{joule}} + 34} = \frac{\dfrac{(2300\ \text{joule})^2}{\text{sec}\cdot\text{joule}} + \dfrac{2300\ \text{joule}}{\text{sec}} - 365\dfrac{\text{joule}}{\text{sec}}}{(4.0)\ \dfrac{(2300\ \text{joule})}{\text{joule}} + 34}$$
$$= 573.09\ \frac{\text{joule}}{\text{sec}}\ (570\ \text{W})$$

The maximum power is 570 W.

LIMITS DO NOT ALWAYS EXIST

When analyzing data, the safety professional needs to be aware that not every function has a limit at every point.

The limit does not exist when the functions, for example, do not overlap, when the functions are divergent, or when they oscillate.

The function $f(x) = |x|/x$ does not approach a limit as x approaches zero. The value of $f(x)$ is -1 if $x < 0$ and $+1$ if $x > 0$. No number L can serve as a limit. The one-sided limit from the left as x approaches zero is -1, and the one-sided limit from the right is $+1$.

For the function $f(x) = 1/x$ as x approaches zero, $f(x)$ does not approach a limit. For this function the limit is positive and unbounded when $x > 0$, but the limit is negative and unbounded when $x < 0$. The one-sided limit as x approaches zero from the positive side is the unbounded value called $+\infty$. The one-sided limit as x approaches zero from the negative side is the unbounded value called $-\infty$.

For the function $f(x) = \sin 1/x$ as x approaches zero there is no limit, because the function oscillates.

SEQUENCES AND PROGRESSIONS

The safety professional will use numerical *sequences* and *progressions* for analyzing, for example, economic aspects of a project such as rates of interest and investment or when studying the amount of a drug delivered into a living organism or the effect of an infectious organism as a disease is developed.

A sequence is an ordered list of numbers that contains a pattern. For example, a sequence of positive even numbers less than 12 is 2, 4, 6, 8, and 10.

Within a sequence each member is called a *term*. The terms are determined by their position in the sequence. For example, in the sequence 1, 3, 5, 7, the first term is 1 and the fourth term is 7. The ellipsis symbol … indicates that the sequence continues beyond the terms that are written to infinity. For example, in the sequence 4, 8, 16, 32, … the next term would be 64, then 128, and so on. Often one of the terms of a sequence of numbers needs to be determined. A way of representing a sequence in general is by using letters and indices; for example, $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$. The sequence is said to be convergent if the limit $L = \lim_{n \to \infty} a_n$ exists and is finite. The sequence diverges if L is infinite or does not exist. For example, the sequence with general term e^n/n diverges.

A progression is a sequence with a specific pattern. The progression may be arithmetic if each term of the sequence is obtained by adding a number to the previous term. The progression is geometric when each term is obtained by multiplying the previous term by a number.

The sequence 1/2, 1/3, 1/4, 1/5, …, $1/n$ is the harmonic sequence. The harmonic sequence converges:

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

When determining the next term in a sequence, determine what the relationship is between the terms of the sequence. Determine then if the sequence is an arithmetic progression, a geometric progression, or a more complex sequence. If the sequence is an arithmetic progression, determine the common value and make sure that each term of the sequence can be generated through the addition of that common value to the preceding term. Similarly, if the sequence is a geometric progression, determine the common value and confirm that each term can be multiplied by this constant value to generate the following term. When determining the value of a specific term, it may be useful to make a little diagram to ensure that you are solving for the correct position.

Example 1.28. The positive numbers, in the following sequence, describe the number of new individuals of a particular type of fungus, and the negative numbers indicate the number of deaths at the end of a period. Determine the number of new individuals at the end of the eighth term of the sequence −1, 2, −2, −4, 8, −32, −256,…

Every number of the sequence is the product of the two immediately preceding numbers. The answer is the product of the sixth term times the seventh term in the sequence. The sixth term is −32. The seventh term is −256. The eighth term is:

$$(-32)(-256) = 8192$$

Example 1.29. A slow-acting drug is injected into the body. The drug dissipates in such a way that the amount of active drug in the body at the end of the sixth period after injection is 22. At the end of each period after the first, the amount of active drug in the body is 3.5 less than the number immediately preceding it. Determine the amount of active drug present in the body at the end of the second period of the sequence.

Because each term is 3.5 less than the previous term, the fifth term is 22 – 3.5 = 25.5, the fourth term is 25.5 – 3.5 = 29, the third term is 29 – 3.5 = 32.5, and the second term is 32.5 – 3.5 = 36. The sequence is 39.5, 36, 32.5, 29, 25.5, 22, . . . The second term of the sequence is 36.

Example 1.30. A disease is transmitted in such a way that average of cells infected is given by the following sequence: 3.200, 7.040, 15.49, 34.07, …What is the next term of the sequence?

Multiplying each term by the constant value of 2.2 generates this sequence. To generate the next term, multiply 34.0736 by 2.2 to give 74.961920 or 74.96.

The next term in the sequence is 74.96.

Example 1.31. Money in a bank account earns a certain amount of interest each period. In this account the amount earned in interest at the end of the fifth period is $86. The interest earned at the end of each period except the first is equal to twice the interest earned in the previous period plus $2. What is the amount earned at the end of the second period?

Any term of the sequence, n, may be calculated as:

$$t_5 = 2t_4 + \$2$$

If the fifth term is $86, then:

$$t_5 = 2t_4 + \$2 = \$86$$

Solving for the interest earned in the fourth term gives:

$$t_4 = \$42$$

The interest earned in the third term can be calculated as follows:

$$t_4 = 2t_3 + \$2$$
$$42 = 2t_3 + \$2$$
$$t_3 = \$20$$

The interest earned in the second term is calculated as:

$$t_3 = 2t_2 + \$2 = \$20$$

Solving:

$$t_2 = \$9$$

The interest earned in the second term is $9.

PRACTICAL PROBLEMS: PIPES

Hazardous materials are often transported from one place to another through pipes. In addition, the design of fire sprinkler systems requires a careful consideration of the flow of liquids in pipes. To answer the questions that arise in these design processes, the safety professional uses the algebra concepts that have been presented so far in this chapter.

Characteristics of Pipes

Generally, the word *pipe* refers to a closed conduit of circular cross section and constant internal diameter. The most common way of transporting a fluid from one point to another is to have the fluid flow through a piping system. Pipes have circular cross sections because circular pipes offer the most structural strength and the greatest cross-sectional area per unit of wall surface compared with other possible cross-sectional shapes. Schedule 40 stainless steel or polyvinyl chloride pipe is commonly used in industries that handle caustic or dangerous materials. Table 1.1 shows some of its characteristics.

The table shows the weight of water per foot of pipe. To determine the weight of a different liquid, the specific gravity (density) of that liquid must be known.

Example 1.32. A fire sprinkler system uses 1-in diameter Schedule 40 pipe. How much will the water from the sprinkler system weigh per foot of pipe?

As the pipe is 1-in nominal diameter Schedule 40 pipe, from Table 1.1, the weight of water per foot of pipe is 0.375 lb.

Example 1.33. A 1.5-in diameter Schedule 40 pipe transports a fluid having a specific gravity of 1.20. What is the mass of fluid in the pipe per foot of pipe?

From Table 1.1, the weight of water, w, per foot of pipe in a 1.5-in nominal diameter Schedule 40 pipe is 0.881 lb/ft. The specific gravity of the fluid, S, is given as 1.20. To determine the weight of the fluid, multiply the weight of water by the specific gravity of the fluid. Therefore, the weight of fluid per foot of pipe is:

$$w_f = wS = \left(0.881\frac{\text{lb}}{\text{ft}}\right)(1.20) = 1.0572\frac{\text{lb}}{\text{ft}}\ (1.06\ \text{lb/ft})$$

where:

w_f is the weight of the fluid per foot of pipe

w is the weight of water per foot of pipe

S is the specific gravity of the fluid

TABLE 1.1. Characteristics of Schedule 40 Pipe

Nominal diameter in	Internal diameter in	Volume of contents (per foot of pipe) in^3/ft	Volume of contents (per foot of pipe) gal/ft	Weight of water (per foot of pipe) lb/ft
1/8	0.269	0.0568	2.459×10^{-4}	0.025
1/4	0.364	0.1041	4.506×10^{-4}	0.045
3/8	0.493	0.1909	8.264×10^{-4}	0.083
1/2	0.622	0.3039	0.00132	0.132
3/4	0.824	0.5333	0.00231	0.231
1	1.049	0.8643	0.00374	0.375
1 1/4	1.380	1.496	0.00648	0.649
1 1/2	1.610	2.036	0.00881	0.882
2	2.067	3.356	0.0145	1.45
2 1/2	2.469	4.788	0.0207	2.07
3	3.068	7.393	0.0320	3.20
3 1/2	3.548	9.887	0.0428	4.29
4	4.026	12.73	0.0551	5.50
5	5.047	20.01	0.0866	8.67
6	6.065	28.89	0.125	12.51
8	7.981	50.03	0.217	21.70
10	10.020	78.85	0.341	34.20
12	11.938	111.93	0.484	48.50
14	13.124	135.28	0.586	58.64
16	15.000	176.72	0.765	76.58
18	16.876	223.68	0.968	96.93
20	18.812	277.95	1.203	120.46
24	22.624	402.00	1.740	174.23

Example 1.34. Benzene is transported in a 2-in diameter stainless steel Schedule 40 pipe from a storage tank to a location 100 ft away. What is the volume of benzene in the pipe?

From Table 1.1, for a 2-in nominal Schedule 40 pipe the volume inside the pipe per foot of pipe, v, is 3.356 in^3/ft. The length of pipe, l, is given as 100 ft. The total volume of benzene inside the pipe is:

$$V_p = lv = (100\,\mathrm{ft})\left(\frac{3.356\,\mathrm{in}^3}{\mathrm{ft}}\right) = 335.6\,\mathrm{in}^3\ (336\ \mathrm{in}^3)$$

Flow of Liquids in Pipes

The safety professional might be involved in the calculation of flow rates of liquids in pipes. *Volumetric flow* is defined as the volume of liquid to pass a certain point per unit of time. *Mass flow* is defined as the mass of liquid to pass a certain point per unit of time.

Density, ρ, of a liquid is defined as the mass of the liquid per unit volume.

Here are some useful equations:

$$V = qt \tag{1.38}$$

where:

V = volume
q = volumetric flow rate
t = time

$$M = kt \tag{1.39}$$

where:

M = mass
k = mass flow rate
t = time

$$Q = \frac{W}{\rho} \tag{1.40}$$

where:

Q = volumetric flow rate
W = mass flow rate
ρ = density

To convert from volumetric flow of a liquid to mass flow of a liquid, the mass per unit volume of the liquid must be known. For example, referring to the Conversion Factor (in the Appendix at the end of this book), one gallon of water has a mass of 8.345 lbs.

Example 1.35. A pump that is capable of discharging water at a rate of 542.5 gal/min feeds a fire sprinkler system. Determine the volume of water discharged in 2.0 hours when the sprinkler system is opened.

The volumetric flow rate of water, q, is given as 542.5 gal/min. The discharge time, t, is given as 2.0 h. The volume of water discharged is:

$$V = qt = \left(542.5\frac{\text{gal}}{\text{min}}\right)(2\text{hr})\left(60\frac{\text{min}}{\text{hr}}\right) = 65,000 \text{ gal}$$

Example 1.36. A pump capable of discharging water at a volumetric flow rate of 326.26 gal/min feeds a fire sprinkler system installed in a building. Assume that 1 gallon of water has a mass of 8.345 pounds. Determine the mass of water discharged in 20 min.

The volumetric flow rate of water, q, is given as 324.26 gal/min. The time, t, is given as 20 min. To convert from volume to mass, multiply the volume by the mass per unit volume. Therefore the mass of water discharged is:

$$V = qt = \left(324.26\frac{\text{gal}}{\text{min}}\right)(20\text{min})$$
$$= 6485.2 \text{ gal}$$

Multiply the volume in gallons by mass of one gallon:

$$6485.2 \text{ gal}\left(8.345\frac{\text{lb}}{\text{gal}}\right) = 54118.994 \text{ lb } (54,000 \text{ lb})$$

Example 1.37. A fuel oil having a density of 58.62 lb/ft^3 at a temperature of 60°F flows through a 2-in Schedule 40 pipe at a mass flow rate of 25,000 lb/hr. What is the volumetric flow rate of the fuel oil in gal/min?

The density, ρ, of the fuel oil is given as 58.62 lb/ft^3. The mass rate, W, is given as 25,000 lb/hr. The conversion from mass flow to volumetric flow is given by the equation below. Since the mass rate is given in pounds per hour, a conversion from hours to minutes will also be necessary:

$$Q = \frac{W}{\rho}$$
$$= \frac{\left(25,000\frac{\text{lb}}{\text{hr}}\right)\left(7.48\frac{\text{gal}}{\text{ft}^3}\right)\left(\frac{\text{hr}}{60 \text{ min}}\right)}{58.62\,\frac{\text{lb}}{\text{ft}^3}}$$
$$= 53.17 \text{ gal/min} \approx 53 \text{ gal/min}$$

because 25,000 has only two (2) significant figures.

GEOMETRY

Routinely, the safety profession will use *geometry* in finding perimeters, areas, and volumes of basic figures, such as rectangles, parallelograms, trapezoids, squares, triangles, circles, prisms and cylinders.

Perimeters

The *perimeter* of a polygon is the sum of all its sides. For example, a triangle with sides 20.0 m, 23.4 m, and 34.0 m has a perimeter of 20.0 m + 23.4 m + 34.0 m = 77.4 m.

The perimeter of a circle is known as the *circumference*. The perimeter of a circle of radius, r, and diameter $2r$ is:

$$2r \text{ or } \pi d \tag{1.41}$$

where, π (pi) is an irrational number, 3.14 to three (3) significant figures.

Example 1.38. A 5-in Schedule 40 pipe transports *n*-hexane. Determine the internal circumference of the pipe.

From Table 1.1, the diameter of the pipe, d, is 5.047 in. The internal circumference of the circle is:

$$p = \pi d = \pi(5.047 \text{ in}) = 15.865 \text{ in}$$

Example 1.39. The lengths of the sides of a rectangular rainwater pond are 28.4 ft and 14.6 ft. Determine the perimeter of the pond.

The length of one of the largest sides of the rectangular pond, a, is given as 28.4 ft, and the shortest side, b, is given as 14.6 ft. Because the pond is rectangular, it has a pair of equal sides equal to a and a pair of equal sides equal to b. The perimeter of the pond is:

$$p = 2a + 2b = (2)(28.4 \text{ ft}) + (2)(14.6 \text{ ft}) = 86.0 \text{ ft}$$

Example 1.40. A regular hexagon-shaped tank having six (6) sides of 12.84 ft each contains a crushed mineral. Determine the perimeter of the tank.

The side of the hexagon, a, is given as 12.84 ft. The perimeter of the hexagon equals 6 times the length of a side of the tank:

$$p = 6a = (6)(12.84 \text{ ft}) = 77.04 \text{ ft}$$

Areas

The *area* of a triangle of height h and base b is:

$$A = \frac{bh}{2} \tag{1.42}$$

The area of a square having side a is:

$$A = a^2 \tag{1.43}$$

The surface area of a cylinder of radius r and length l is:

$$A = 2\pi rl \tag{1.44}$$

The area of a circle having a radius r is:

$$A = \pi r^2 \tag{1.45}$$

The area of a semicircle having a radius r is:

$$A = \frac{1}{2}\pi r^2 \tag{1.46}$$

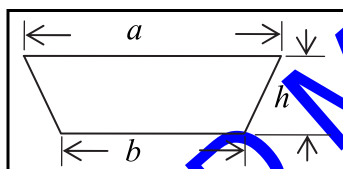

FIGURE 1.1. This is a graphical representation of the area of a trapezoid.

The area of a *trapezoid* (Figure 1.1) with parallel sides a and b and height h is:

$$A = \frac{(a + b)h}{2} \tag{1.47}$$

The area of a triangle having a base b and a height h is:

$$A = \frac{bh}{2} \tag{1.48}$$

The area of a rectangle having sides a and b is:

$$A = ab \tag{1.49}$$

Example 1.41. A section of a roof is a triangle with side 28.9 ft and height 10.2 ft. The section is covered with asbestos cement roof shingles. To avoid damaging the shingles and minimize production of dust containing asbestos fibers, each individual shingle is to be removed. For safety reasons, two (2) qualified workers are to perform the task. The area of one (1) shingle was determined to be 3.7 ft^2. How many shingles will be removed?

The side of the roof, a, is given as 28.9 ft. The height of the roof, h, is given as 10.2 ft. The area of the roof is:

$$A_r = \frac{ah}{2} = \frac{(28.9\text{ ft})(10.2\text{ ft})}{2} = 147.39\text{ ft}$$

The area of each shingle, A_s, is given as 3.7 ft^2. The number of shingles to be removed is:

$$N = \frac{A_r}{A_s} = \frac{147.39\text{ ft}}{3.7\text{ ft}^2} = 39.835\text{ shingles (40 shingles)}$$

Example 1.42. In a refinery, a rectangular sump is to be designed so that the length is 50 ft and the width is 34 ft. A walk of uniform width around it is to be designed with an area not greater than 3540 ft^2. Determine the width of the walk.

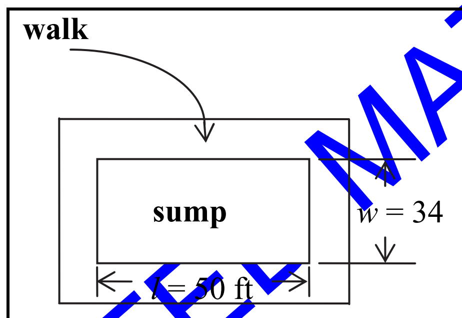

FIGURE 1.2. This is a graphical representation of the dimensions of rectangular sump and walk from Example 1.42.

The length, l, of the sump is given as 50 ft. The width, w, of the sump is given as 34 ft. The maximum area of the walk, S, is given as 3540 ft^2. Let the maximum width of the walk be x. The walk is made of four (4) rectangles as shown in Figure 1.3.

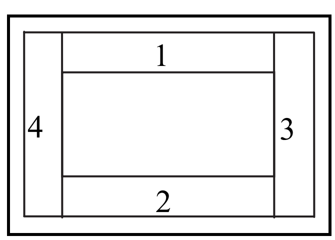

FIGURE 1.3. This is a graphical representation of the walk area decomposed into rectangles.

Rectangles 1 and 2 are of the same size. Rectangles 3 and 4 are of the same size. Each of rectangles 1 and 2 has a length equal to the length of the sump, which is 50 ft. The width of each of the rectangles 1 and 2 is x. The area of each of rectangles 1 or 2 is:

$$A_1 = A_2 = lx \tag{1.50}$$

From Figure 1.3, the length of each of rectangles 3 and 4 is:

$$l_3 = l_4 = w + 2x \tag{1.51}$$

Because the walk is of the same width, the width of each of rectangles 3 and 4 is x. The area of each of the rectangles 3 or 4 is:

$$\begin{aligned} A_3 = A_4 &= l_3 w = l_4 w \\ &= (w + 2x)w = w^2 + 2xw \end{aligned} \tag{1.52}$$

The total area (S) is 2 times Equation 1.49 plus 2 times Equation 1.52:

$$S = 2lx + 2(w^2 + 2xw)$$

Solving the preceding equation gives:

$$\begin{aligned} X = \frac{S - 2w^2}{4w + 2l} &= \frac{3540\ \text{ft}^2 - (2)(34\ \text{ft})^2}{(4)(34\ \text{ft}) + (2)(50\ \text{ft})} \\ &= 5.20\ \text{ft}\ (5.2\ \text{ft}) \end{aligned}$$

Volumes

The volume of a cylinder of radius r and length l is:

$$V = \pi r^2 l \tag{1.53}$$

The volume of the sphere of radius r is:

$$V = \frac{4\pi r^3}{3} \tag{1.54}$$

Example 1.43. A liquid pesticide label indicates that 250 mL of pesticide will be sufficient to spray a surface of 100 m^2. Determine the volume of pesticide required to spray a square field having a side of 120 m.

The side of the field, a, is given as 120 m. The surface area of the field is:

$$A = a^2 = (120\ \text{m})^2 = 14,400\ \text{m}^2$$

The volume of pesticide, V, per unit area is given as 250 mL/100 m^2. The total volume of pesticide required is:

$$V_t = AV = (14,400\ \text{m}^2)\left(\frac{250\ \text{mL}}{100\ \text{m}^2}\right)\left(\frac{\text{L}}{1000\ \text{mL}}\right)$$
$$= 36\ \text{L}$$

Example 1.44. A cylindrical quartz cell of radius 0.50 cm and length 4.0 cm is to be used in an infrared instrument to analyze a liquid sample of a pollutant. Determine the volume of the sample required to fill the infrared cell.

The radius of the cell, r, is given as 0.50 cm. The length of the cell, l, is given as 4.0 cm. The volume of the cell is:

$$V = \pi r^2 l = \pi(0.50\ \text{cm})^2(4.0\ \text{cm})$$
$$= 3.142\ \text{cm}^3\ (3.1\ \text{cm}^3)$$

Example 1.45. A spherical pressurized container is to be designed to store the refrigerant Freon. At a set of temperature and pressure conditions the volume of Freon to be stored is 1,200,345 ft^3. Determine the radius of the container.

The required volume, V, is 1,200,345 ft^3. The equation relating radius and volume is:

$$V = \frac{4\pi r^3}{3}$$

Solving the preceding equation for the radius of the container gives:

$$r = \left(\frac{3V}{4\pi}\right)^{1/3} = \left(\frac{(3)(1,200,345\ \text{ft}^3)}{4\pi}\right)^{1/3}$$
$$= 65,928\ \text{ft}\ (66\ \text{ft})$$

CONCLUSION

The use of symbols instead of numbers is the basis of algebra and allows the generalization of specific solutions to groups of problems. The invention of algebra was essential to mankind's progress in science and engineering. The abstraction that took place in going from tick marks on a rock to a symbol that stood for the tick marks was the first step. The invention of a set of ten (10) symbols that could be used to write *any* number was the second step. The use of letters as abstract symbols to stand for kinds of number that take values by the rules of algebra is the third step.

When the safety professional grasps these facts, he/she can appreciate how each equation is a concise and beautiful statement of how certain things are related.

Algebra allows the solving of many problems as well as allowing proof that some problems cannot be solved, thus saving untold hours of wasted effort.

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